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The Structure of $\bar{I} \otimes \Lambda^n$ in Generic Representation Theory

Geoffrey M. L. Powell*, †

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903

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This paper pursues a study of the category $\mathcal{F}$ of functors between $\mathbb{F}_2$-vector spaces. It is proved that the objects $\bar{I} \otimes \Lambda^n$ occur in short exact sequences $\bar{K}_n \to \bar{I} \otimes \Lambda^n \to \bar{K}_{n-1}$, where each $\bar{K}_n$ is a non-finite functor, which has the property that every proper sub-functor is finite. Moreover, every sub-functor of $\bar{I} \otimes \Lambda^n$ either contains $\bar{K}_n$ or is finite. © 1997 Academic Press

1. INTRODUCTION

The purpose of this paper is to provide a new and conceptual proof that the functors $\bar{I} \otimes \Lambda^n$ in the category $\mathcal{F}$ of *generic representations* are Artinian, a result which was first proved by Piriou in his thesis [8]. Throughout the paper, $\mathbb{F}$ is the prime field with two elements and $\mathcal{F}$ is the category of functors $\mathcal{E}_f \to \mathcal{E}$, where $\mathcal{E}$ denotes the category of $\mathbb{F}$-vector spaces and $\mathcal{E}_f$ the full sub-category of finite-dimensional spaces. $\mathcal{F}$ is an abelian category, with the usual notion of simple object, composition series, socle series, and the like. In particular, an object is said to be *finite* if it has a composition series of finite length and is *Artinian* if every descending sequence of sub-functors stabilizes. The full sub-category of $\mathcal{F}$ having objects which are the colimit of their finite sub-objects is denoted by $\mathcal{F}_\omega$ and is termed the category of *analytic functors*. This category is of interest to topologists due to the connection with the category of unstable modules over the Steenrod algebra, which was established in [4] and studied in [5, 6, 7]. These papers serve as standard references for the material in this paper, along with the book [12].

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† E-mail: gp3e@virginia.edu.

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Let I denote the injective in the category $\mathcal{F}$ which is defined by the natural isomorphism $\mathrm{Hom}_{\mathcal{F}}(F, I) \cong DF(\mathbb{F})$, for every functor $F \in \mathcal{F}$, where $D : \mathcal{F} \to \mathcal{F}^{op}$ denotes the duality functor $DF(V) := F(V^*)^*$, with the star denoting the vector space dual. I decomposes as $\mathbb{F} \oplus \bar{I}$ and the functor $\bar{I}$ is uniserial with socle series given by $(\mathrm{soc}_n/\mathrm{soc}_{n-1})\bar{I} \cong \Lambda^n$, the nth exterior power functor. The sub-functor $\mathrm{soc}_n(\bar{I})$ is denoted by $p_n\bar{I}$, since this corresponds to the polynomial filtration of $\bar{I}$.

The functor $\bar{I} \otimes \Lambda^n$ is the tensor product in $\mathcal{F}$ of $\bar{I}$ with the nth exterior power functor; the interest in this functor arises from the study of $\bar{I} \otimes \bar{I}$, which is filtered by the functors $\bar{I} \otimes p_n\bar{I}$. The sub-quotients of this filtration are the functors $\bar{I} \otimes \Lambda^n$. The results proved here are used in an essential way in the author's paper [9], in which it is proved that the functor $\bar{I}^{\otimes 2}$ is Artinian. This is the first non-trivial case to be established of the Artinian conjecture, due to Lionel Schwartz in conjunction with Nick Kuhn and Jean Lannes.

There is a unique non-trivial map $\overline{\phi}_n : \bar{I} \otimes \Lambda^n \to \bar{I} \otimes \Lambda^{n-1}$, for $n \geq 1$. This map may be described as the composite

$$\bar{I} \otimes \Lambda^n \xrightarrow{\bar{I} \otimes \psi_n} \bar{I} \otimes \Lambda^1 \otimes \Lambda^{n-1} \xrightarrow{\mu \otimes \Lambda^{n-1}} \bar{I} \otimes \Lambda^{n-1},$$

where $\psi_n : \Lambda^n \to \Lambda^1 \otimes \Lambda^{n-1}$ denotes the coproduct map and $\mu : \bar{I} \otimes \Lambda^1 \to \bar{I}$ is given by the product map $I \otimes I \to I$. It is not difficult to see that the composite $\overline{\phi}_{n-1}\overline{\phi}_n$ is zero; more is true:

PROPOSITION 1.1. *The complex* $(\bar{I} \otimes \Lambda^n, \overline{\phi}_n)$, *for* $n \geq 0$, *is an exact chain complex.*

This result appears in a preprint derived from [8]. The exactness may be shown in a number of ways; a proof based on the homology of the Koszul sequences is given in the Appendix.

Notation 1.2. Write $\overline{K}_n$ for the kernel $\mathrm{Ker}\,\overline{\phi}_n$, so that $\overline{K}_0 \cong \bar{I}$ and, for $n \geq 1$, there are short exact sequences of functors $\overline{K}_n \to \bar{I} \otimes \Lambda^n \to \overline{K}_{n-1}$.

The main result of this paper is:

THEOREM 1. 1. *Every proper sub-functor of* $\overline{K}_n$ *is finite, for* $n \geq 0$.

2. *If* F *is a sub-functor of* $\bar{I} \otimes \Lambda^n$, $n \geq 1$, *either* F *is finite or* F *contains* $\overline{K}_n$.

This has the straightforward corollary:

COROLLARY 1.3. *The functor* $\bar{I} \otimes p_n\bar{I}$ *are Artinian.*

Remark 1.4. The theorem may be interpreted as stating that the functors $\overline{K}_n$ are simple in the category $\mathcal{F}$ localized away from the finite functors and that the extension $\overline{K}_n \to \bar{I} \otimes \Lambda^n \to \overline{K}_{n-1}$ does not split in the localized category.

The key to the arguments in the paper is the use of an endo-functor $\tilde{\nabla}$ which the author introduced in [10] had has exploited in [9, 11]. These new techniques have proved to be a powerful tool and help to give a natural generalization of the idea of polynomial filtrations.

2. BACKGROUND ON THE CATEGORY $\mathcal{F}$

The category $\mathcal{F}$ is equipped with an exact endo-functor, $\Delta : \mathcal{F} \to \mathcal{F}$, called the *difference functor*. This may be defined by $(\Delta F)(V) := F(V \oplus \mathbb{F})/F(V)$, where $F(V) \hookrightarrow F(V \oplus \mathbb{F})$ is the canonical (split) inclusion. The functor Δ is both a left adjoint and right adjoint; it is left adjoint to the functor $- \otimes \bar{I}$ and is right adjoint to the functor $- \otimes D\bar{I}$. A basic result is the following behaviour with respect to the tensor product of functors:

$$\Delta(F \otimes G) \cong (\Delta F \otimes G) \oplus (F \otimes \Delta G) \oplus (\Delta F \otimes \Delta G).$$

There is an isomorphism $\Delta\Lambda^n \cong \Lambda^{n-1}$ for $n \geq 1$, where Λ^0 is taken to be the constant functor $\mathbb{F}$ and $\Delta\mathbb{F} = 0$; similarly $\Delta\bar{I} \cong I$.

The exponential property of the symmetric powers and exterior powers is required; this is treated in [2, 5] and rests on ideas in [3] using the category of bi-functors. For the purposes of this paper, the following is sufficient: if S^k denotes the kth symmetric power, then for F, G functors in $\mathcal{F}$, there are isomorphisms:

$$\mathrm{Hom}_{\mathcal{F}}(F \otimes G, \Lambda^n) \cong \bigoplus_{j=0}^{n} \mathrm{Hom}_{\mathcal{F}}(F, \Lambda^j) \otimes \mathrm{Hom}_{\mathcal{F}}(G, \Lambda^{n-j})$$

$$\mathrm{Hom}_{\mathcal{F}}(F \otimes G, S^n) \cong \bigoplus_{j=0}^{n} \mathrm{Hom}_{\mathcal{F}}(F, S^j) \otimes \mathrm{Hom}_{\mathcal{F}}(G, S^{n-j}).$$

This allows the proof of the following result:

LEMMA 2.1. *The functor $\bar{I} \otimes \Lambda^n$, for $n \geq 0$, has no finite quotient.*

Proof. Suppose that G is finite and there is a surjection $\bar{I} \otimes \Lambda^n \twoheadrightarrow G$. The embedding theorem of Kuhn [5] implies that there exists a non-trivial map $G \to S^k$, for some k. Thus, there is a non-trivial composite $\bar{I} \otimes \Lambda^n \twoheadrightarrow G \to S^k$. The exponential property for the symmetric powers implies that $\mathrm{Hom}_{\mathcal{F}}(\bar{I} \otimes \Lambda^n, S^k) \cong \bigoplus_{j=0}^{k} \mathrm{Hom}_{\mathcal{F}}(\bar{I}, S^j) \otimes \mathrm{Hom}_{\mathcal{F}}(\Lambda^n, S^{k-j})$. However, $\bar{I}$ is uniserial and is not finite; hence $\mathrm{Hom}_{\mathcal{F}}(\bar{I}, S^j) = 0$ for any j, providing a contradiction. ∎

This allows the proof of the following result:

LEMMA 2.2. *There is a unique non-trivial map* $\bar{I} \otimes \Lambda^n \to \bar{I} \otimes \Lambda^{n-1}$.

Proof. By adjunction, a non-trivial map $\bar{I} \otimes \Lambda^n \to \bar{I} \otimes \Lambda^{n-1}$ corresponds to a non-trivial map $\Delta(\bar{I} \otimes \Lambda^n) \cong (I \otimes \Lambda^n) \oplus (\bar{I} \otimes \Lambda^{n-1})^{\oplus 2} \oplus \Lambda^{n-1} \to \Lambda^{n-1}$; by the above lemma, this is unique, namely the projection from $\Delta(\bar{I} \otimes \Lambda^n)$ onto the direct summand Λ^{n-1}. ∎

2.1. *The Functor* $\tilde{\nabla}$

The difference functor Δ admits a polynomial filtration $[p_n\Delta]$, for $n \geq 1$, which the author defined and studied in [10]; these functors are right adjoints and are therefore left exact. In applications [9, 11], it has proved invaluable to study the quotients of this filtration, namely the functors $\tilde{\nabla}_n := \Delta/[p_{n-1}\Delta]$, for $n \geq 2$. For the purposes of this paper, it suffices to study the functor $\tilde{\nabla}_2$, which will henceforth be denoted simply by $\tilde{\nabla}$. For a full discussion of these functors, the reader is referred to [10]; an explicit description of $\tilde{\nabla}$ is given as follows.

The dual $D\bar{I}$ of the functor $\bar{I}$ is the functor defined by $D\bar{I}(V) \cong \overline{\mathbb{F}[V]}$, the augmentation ideal of the group ring on the elementary abelian 2-group V. Thus, there is a natural product map $D\bar{I} \otimes D\bar{I} \to D\bar{I}$. The difference functor Δ is right adjoint to $- \otimes D\bar{I}$; an application of Yoneda's lemma shows that the product induces a natural map, for any $F \in \mathcal{F}$:

$$\Delta F \to \Delta^2 F.$$

This may be thought of as a direct summand of the map $F(V \oplus \mathbb{F}) \to F(V \oplus \mathbb{F}^2)$, given by $\sum_{\alpha \in \mathbb{F}^2} F(1_V \oplus \alpha)$, where the sum is taken over $\mathbb{F}^2$ as a set, and an element is identified with a linear map $\mathbb{F} \to \mathbb{F}^2$.

The functor $\tilde{\nabla}F$ is defined to be the image of this map; it should be stressed immediately that the functor $\tilde{\nabla}$ is *not* exact. $\tilde{\nabla}$ is essentially the quotient of the difference functor Δ by the linear part $[p_1\Delta]$; this observation makes the proof of the following a basic exercise:

LEMMA 2.3. 1. $\tilde{\nabla}I = 0$.

2. *If* $m, n \geq 1$, *then* $\tilde{\nabla}(\Lambda^m \otimes \Lambda^n) \cong \Lambda^{m-1} \otimes \Lambda^{n-1}$.
3. $\tilde{\nabla}(\bar{I} \otimes \Lambda^n) \cong \Lambda^{n-1} \oplus (\bar{I} \otimes \Lambda^{n-1})$.

The *simple objects* of $\mathcal{F}$ are indexed by partitions which are 2-regular; namely by finite sequences of positive integers $(\lambda_1 > \cdots > \lambda_s)$, for some s; this is shown in [6]. This correspondence may be chosen so that Λ^n is indexed by (n) and the simple object F_λ indexed by the partition λ occurs as the top factor of the Weyl module functor which is a sub-functor of

$\Lambda^{\lambda_1} \otimes \cdots \otimes \Lambda^{\lambda_s}$; see [1, 6]. The composition factors of the functors $\bar{I} \otimes \Lambda^n$ are all of the form $F_{(\lambda_1, \lambda_2)}$, for $\lambda_1 > \lambda_2 > 0$ or $\Lambda^k \cong F_{(k,0)}$.

The basic properties of the functor $\tilde{\nabla}$ which are required for this paper are summarized below:

PROPOSITION 2.4 [10]. 1. *The functor* $\tilde{\nabla} : \mathcal{F} \to \mathcal{F}$ *preserves injections and surjections*.

2. $\tilde{\nabla}\Lambda^k = 0$.

3. *If* $\lambda_1 > \lambda_2 > 0$, *then* $\tilde{\nabla}F_{(\lambda_1, \lambda_2)} \cong F_{(\lambda_1 - 1, \lambda_2 - 1)}$.

This result is the key to all the arguments, since the fact that $\tilde{\nabla}$ preserves injections and surjections implies that, if Q is a sub-quotient of a functor F, then $\tilde{\nabla}Q$ occurs as a sub-quotient of $\tilde{\nabla}F$. In particular, if $F_{(\lambda_1, \lambda_2)}$ with $\lambda_2 > 0$ occurs as a composition factor of F, then $F_{(\lambda_1 - 1, \lambda_2 - 1)}$ occurs as a composition factor of $\tilde{\nabla}F$.

To perform the inductive proof, the operation of $\tilde{\nabla}$ on the functors $\overline{K}_n$ is required:

PROPOSITION 2.5. *For* $n \geq 1$, $\tilde{\nabla}\overline{K}_n \cong \overline{K}_{n-1}$; $\tilde{\nabla}\overline{K}_0 = 0$.

Proof. $\overline{K}_0 \cong \bar{I}$, so the last statement is clear. Now consider the factorization of $\overline{\phi}_{n+1}$ (for $n \geq 1$) as: $\bar{I} \otimes \Lambda^{n+1} \twoheadrightarrow \overline{K}_n \hookrightarrow \bar{I} \otimes \Lambda^n$. Applying $\tilde{\nabla}$ yields the following composite $\tilde{\nabla}(\bar{I} \otimes \Lambda^{n+1}) \twoheadrightarrow \tilde{\nabla}\overline{K}_n \hookrightarrow \tilde{\nabla}(\bar{I} \otimes \Lambda^n)$. By Proposition 2.4, the first map is surjective; Lemma 2.3 identifies the above as maps $(\bar{I} \otimes \Lambda^n) \oplus \Lambda^n \twoheadrightarrow \tilde{\nabla}\overline{K}_n \hookrightarrow (\bar{I} \otimes \Lambda^{n-1}) \oplus \Lambda^{n-1}$. The restriction of $\tilde{\nabla}\overline{\phi}_{n+1}$ to the summand $\bar{I} \otimes \Lambda^n$ is non-trivial, hence identifies with $\overline{\phi}_n$, using the uniqueness of $\overline{\phi}_n$ and the fact that $\bar{I} \otimes \Lambda^n$ has no non-finite quotients. Equally, the map restriction of $\tilde{\nabla}\overline{\phi}_n$ to Λ^n may be seen to be the canonical inclusion $\Lambda^n \hookrightarrow \bar{I} \otimes \Lambda^{n-1}$, which lies in $\overline{K}_{n-1} \subset \bar{I} \otimes \Lambda^{n-1}$. It follows that the image of $\tilde{\nabla}\overline{\phi}_n$ is $\overline{K}_{n-1}$, as required. ∎

3. PROOF OF THE THEOREM

The proof of the theorem is based on the following result, which is proved in Section 4:

PROPOSITION 3.1. 1. *Suppose that* $F \subset \bar{I} \otimes \Lambda^n$ *is a sub-functor, with* $n \geq 1$, *such that* F *is not finite. Then* F *has an infinite number of composition factors of the form* $F_{(\lambda_1, \lambda_2)}$, *where* $\lambda_1 > \lambda_2 > 0$.

2. *Suppose that* $n > 1$ *and that* $\bar{I} \otimes \Lambda^n \to G$ *is a non-trivial surjection. Then* G *is not finite and has an infinite number of composition factors of the form* $F_{(\lambda_1, \lambda_2)}$, *where* $\lambda_1 > \lambda_2 > 0$.

This permits the proof of the theorem:

Proof of Part 1. The proof is by induction, with hypothesis that every proper sub-functor of $\bar{K}_n$, with $n \leq N$, is finite. The initial case, $N = 0$, corresponds to the functor $\bar{K}_0$, which is uniserial and therefore has the required property.

Suppose that $F \hookrightarrow \bar{K}_{N+1}$ is a sub-functor, which is not finite; the first part of Proposition 3.1 implies that F has an infinite number of composition factors which are not exterior powers. Therefore, by Proposition 2.4, it follows that $\tilde{\nabla} F \subset \tilde{\nabla} \bar{K}_{N+1} \cong \bar{K}_N$ is not finite. The inductive hypothesis therefore implies that $\tilde{\nabla} F \cong \bar{K}_N$.

Write G for the cokernel of $F \hookrightarrow \bar{K}_{N+1}$; the induced sequence of maps $\tilde{\nabla} F \to \tilde{\nabla} K_{N+1} \twoheadrightarrow \tilde{\nabla} G$ shows that $\tilde{\nabla} G = 0$, since the composite is zero, the second map is a surjection, and the first map is an isomorphism.

The composite $\bar{I} \otimes \Lambda^{N+2} \twoheadrightarrow \bar{K}_{N+1} \twoheadrightarrow G$ is a surjection, with $N + 1 \geq 2$, so one may appeal to the second statement of Proposition 3.1. If G were non-trivial, then it would have an infinite number of composition factors of the form $F_{(\lambda_1, \lambda_2)}$, with $\lambda_1 > \lambda_2 > 0$. In particular, one would have $\tilde{\nabla} G \neq 0$, by Proposition 2.4. Conclude that $G = 0$, so that $F = \bar{K}_n$, as required. ∎

Remark 3.2. Piriou [8] proves a slightly more explicit result concerning the structure of the functors $\bar{K}_n$; namely that, for all $k \geq 1$, $n \geq 1$, there exists an explicit $N = N(k, n)$ such that the presence of a composition factor of polynomial degree k in a sub-functor F of $\bar{K}_n$ implies that the largest sub-functor of $\bar{K}_n$ of polynomial degree N is contained in F. It is not difficult to refine the above argument to recover this result.

The advantage of the approach to the result presented here is that the techniques fit naturally into the work of [9] and have certain generalizations; this is not the case for the proof in [8], which is longer and relies heavily on classical techniques in the representation theory of the general linear groups.

Proof of Part 2. Suppose that $n \geq 1$ and $t \geq 0$; write $K_{n,t}$ for the image of $\bar{K}_n$ under the quotient map $\bar{I} \otimes \Lambda^n \to (\bar{I}/p_t \bar{I}) \otimes \Lambda^n$ and $Q_{n,t}$ for the quotient of the inclusion $K_{n,t} \hookrightarrow (\bar{I}/p_t \bar{I}) \otimes \Lambda^n$; thus $Q_{n,t}$ is a quotient of $\bar{K}_{n-1}$. The statement of the theorem is equivalent to the assertion that the extension

$$K_{n,t} \to \left(\bar{I}/p_t \bar{I}\right) \otimes \Lambda^n \to Q_{n,t} \tag{1}$$

is not split for any $t \geq 0, n \geq 1$. (This may be seen by applying the first statement of the theorem and using the fact that the kernel of the surjection $\bar{I} \otimes \Lambda^n \to (\bar{I}/p_t \bar{I}) \otimes \Lambda^n$ is $p_t \bar{I} \otimes \Lambda^n$, which is a finite functor).

The proof is by induction upon n; the case $n = 1$ is treated by observing that $Q_{1,t}$ is a non-trivial quotient of $\bar{I}$; if $Q_{1,t}$ were a sub-functor of $(\bar{I}/p_t\bar{I}) \otimes \Lambda^1$, then the pull-back of $Q_{1,t}$ to a sub-functor of $\bar{I} \otimes \Lambda^1$ would violate the first statement of Proposition 3.1.

The induction is achieved as in Part 1; if $t > 1$, then $\tilde{\bar{\nabla}}$ applied to the exact sequence (1) identifies with the exact sequence:

$$K_{n-1,t-1} \to \left(\bar{I}/p_{t-1}\bar{I}\right) \otimes \Lambda^{n-1} \to Q_{n-1,t-1}. \tag{2}$$

This may be shown by first establishing the middle term, using the techniques used earlier, and then identifying the outer terms, which are defined as the images of well-understood maps.

A splitting of (1) would imply a splitting of (2), which does not exist, by induction. This establishes the inductive step if $t > 1$. The final case $t = 1$ may be treated in the same way, except that one has an extra factor of Λ^{n-1} which occurs in the sequence. ∎

4. PROOF OF THE KEY PROPOSITION

The purpose of this section is to prove Proposition 3.1, which is the key argument in the proof of the theorem. The proof rests on the following simple technical lemmas.

For notational simplicity, the sub-quotient $p_{n+1}\bar{I}/p_{n-1}\bar{I}$ of $\bar{I}$ is written as $\mathsf{E}(n)$; it lies in a short exact sequence $\Lambda^n \to \mathsf{E}(n) \to \Lambda^{n+1}$. It is an elementary fact that, for $n \geq 2$, $\Delta\mathsf{E}(n) = \mathsf{E}(n-1)$, whereas, for $n = 1$, $\Delta\mathsf{E}(1) = \mathbb{F} \oplus \Lambda^1$. Recall that D denotes the duality functor $D : \mathsf{F} \to \mathsf{F}^{op}$, which was defined in the Introduction; D commutes with tensor products and $D\Lambda^n \cong \Lambda^n$. Thus, $D\mathsf{E}(n)$ occurs as an extension: $\Lambda^{n+1} \to D\mathsf{E}(n) \to \Lambda^n$, for $n \geq 1$. These sequences represent the only non-trivial extensions between exterior powers; one proof of the following is given in [2]:

LEMMA 4.1. *Suppose that $m, n \geq 1$ are integers; then*

$$\operatorname{Ext}^1_{\mathsf{F}}(\Lambda^m, \Lambda^n) \cong \begin{cases} \mathbb{F}, & |m-n| = 1 \\ 0, & \textit{otherwise}. \end{cases}$$

The next lemma admits a "dual statement," obtained by applying the functor D.

LEMMA 4.2. *Suppose that $a, b \geq 1$ are integers.*

1. *Λ^m embeds in $\Lambda^a \otimes \Lambda^b$ if and only if $m = a + b$.*

2. *If* $m \geq 2$, *then*

$$\operatorname{Hom}_{\mathcal{F}}\big(DE(n), \Lambda^a \otimes \Lambda^b\big) \cong \begin{cases} \mathbb{F}, & m = a + b \\ 0, & \text{otherwise}. \end{cases}$$

3. $\operatorname{Hom}_{\mathcal{F}}(E(a+b), \Lambda^a \otimes \Lambda^b) = 0.$

Proof. The first statement is well known and follows by the exponential property; there is a unique non-trivial map $\Lambda^{a+b} \hookrightarrow \Lambda^a \otimes \Lambda^b$, which is given by the coproduct.

For the second statement, for $a + b = m$, there is the unique non-trivial map $DE(m) \twoheadrightarrow \Lambda^m \hookrightarrow \Lambda^a \otimes \Lambda^b$. By the first statement, it remains to show that $\operatorname{Hom}_{\mathcal{F}}(DE(a+b-1), \Lambda^a \otimes \Lambda^b)$ is zero; any such non-trivial map would have to be injective.

One proves by induction on $a + b$ that no such injections exist: for $m = 2, a + b = 3$, one checks the case $\Lambda^2 \otimes \Lambda^1$. This functor decomposes as $\Lambda^3 \oplus F_{(2,1)}$, so the result holds here. Now suppose the result true for $a + b < n$ and suppose that there exists an injection $f: DE(n-1) \hookrightarrow \Lambda^a \otimes \Lambda^{n-a}$. Applying $[p_1\Delta]$ gives an injection: $DE(n-2) \hookrightarrow (\Lambda^{a-1} \otimes \Lambda^{n-1}) \oplus (\Lambda^a \otimes \Lambda^{n-a-1})$. The induction hypothesis shows that no such map exists.

For the final part, use the fact that Λ^{a+b+1} is not a composition factor of $\Lambda^a \otimes \Lambda^b$. A direct proof of this is given by observing that $\Delta^{a+b+1}\Lambda^{a+b+1} = \mathbb{F}$, whereas $\Delta^{a+b+1}(\Lambda^a \otimes \Lambda^b) = 0$. ∎

LEMMA 4.3. *Suppose that* $a \geq 1$, $b \geq 1$ *are integers. Then* $\operatorname{Hom}_{\mathcal{F}}(E(a+b), E(a) \otimes \Lambda^b) = 0$.

Proof. The proof is by a double induction on increasing a, b. The exponential property implies immediately that $\operatorname{Hom}_{\mathcal{F}}(\Lambda^{a+b+1}, E(a) \otimes \Lambda^b) = 0$, so that any non-trivial map in this Hom-set must be injective. To start the induction, one considers the two cases $a = 1$ and $b = 1$.

$a = 1$, $b \geq 2$. Suppose that $f: E(b+1) \hookrightarrow E(1) \otimes \Lambda^b$ is an injection. Apply the functor $[p_1\Delta]$; this gives an injection $[p_1\Delta]f: E(b) \hookrightarrow \{E(1) \otimes \Lambda^{b-1}\} \oplus \{\Lambda^b \oplus (\Lambda^1 \otimes \Lambda^b)\}$. Induction on b implies that $\operatorname{Hom}_{\mathcal{F}}(E(b), E(1) \otimes \Lambda^{b-1}) = 0$ and it is immediate that a map in $\operatorname{Hom}_{\mathcal{F}}(E(b), \Lambda^b \oplus (\Lambda^1 \otimes \Lambda^b))$ is not injective, since $\Lambda^1 \otimes \Lambda^b$ does not contain a factor of Λ^b in its socle.

$a \geq 1$, $b = 1$. Any map $g: E(a+1) \hookrightarrow E(a) \otimes \Lambda^1$ extends to an injection: $E(a+1) \hookrightarrow E(a) \otimes \bar{I}$. Using the adjunction isomorphism (Δ is left adjoint to $\otimes \bar{I}$), there is a unique such non-trivial map. This map factorizes as ϕ through $E(a) \otimes S^2$, where $S^2 = p_2\bar{I}$ is the second symmetric power; the map may be described explicitly in terms of maps derived from the coproduct map $\bar{I} \to \bar{I} \otimes \bar{I}$. In particular, it fits into the commutative diagram

$$\begin{array}{ccc}
p_{a+2}\bar{I} & \rightarrow & \left(\bar{I}/p_{a-1}\bar{I}\right) \otimes \bar{I} \\
\downarrow & & \uparrow \\
\mathsf{E}(a+1) & \xrightarrow{\phi} & \mathsf{E}(a) \otimes S^2 \\
\Big| & & \downarrow \\
\Big| & & (\Lambda^{a+1} \otimes S^2) \oplus \left(\mathsf{E}(a) \otimes \Lambda^2\right) \\
\downarrow & & \uparrow \\
\Lambda^{a+2} & \rightarrow & (\Lambda^{a+1} \otimes \Lambda^1) \otimes (\Lambda^a \otimes \Lambda^2),
\end{array}$$

where the top and bottom rows are induced by coproduct maps and the vertical maps are natural projections or inclusions (for example those induced by maps in the short exact sequence $\Lambda^1 \to S^2 \to \Lambda^2$). In particular, the composite of ϕ with the projection $\mathsf{E}(a) \otimes S^2 \to \mathsf{E}(a) \otimes \Lambda^2$ is non-trivial, so that ϕ cannot factor through $\mathsf{E}(a) \otimes \Lambda^1$.

Inductive step for $a > 1$, $b > 1$. This step is similar to the case $a = 1$; suppose that f is a non-trivial map $f: \mathsf{E}(a+b) \hookrightarrow \mathsf{E}(a) \otimes \Lambda^b$ and apply $[p_1\Delta]$ to get

$$[p_1\Delta]f: \mathsf{E}(a+b-1) \hookrightarrow \left(\mathsf{E}(a-1) \otimes \Lambda^b\right) \oplus \left(\mathsf{E}(a) \otimes \Lambda^{b-1}\right).$$

This is not possible, since $\mathrm{Hom}_{\mathsf{F}}(\mathsf{E}(a+b-1), \{\mathsf{E}(a-1) \otimes \Lambda^b\} \oplus \{\mathsf{E}(a) \otimes \Lambda^{b-1}\})$ is zero, by induction. ∎

Lemma 4.4. *Suppose that* $a \geq 1$, $b > 1$ *are integers. Then* $\mathrm{Hom}_{\mathsf{F}}(\mathsf{E}(a) \otimes \Lambda^b, \mathsf{E}(a+b)) = 0$.

Proof. The proof is again by a double induction on increasing a, b. An argument with the exponential property of Λ implies that any such non-trivial maps are *surjective*.

$a = 1$, $b > 2$. This case may be treated exactly as the case $a = 1$ of the lemma above, but using the functor Δ in place of $[p_1\Delta]$.

$a \geq 1$, $b = 2$. The short exact sequence $\Lambda^1 \to S^2 \to \Lambda^2$ induces an exact sequence:

$$\begin{aligned}
0 &\to \mathrm{Hom}_{\mathsf{F}}\left(\mathsf{E}(a) \otimes \Lambda^2, \mathsf{E}(a+2)\right) \to \mathrm{Hom}_{\mathsf{F}}\left(\mathsf{E}(a) \otimes S^2, \mathsf{E}(a+2)\right) \\
&\to \mathrm{Hom}_{\mathsf{F}}\left(\mathsf{E}(a) \otimes \Lambda^1, \mathsf{E}(a+2)\right).
\end{aligned}$$

The surjection $\Lambda^1 \otimes \Lambda^1 \to S^2$ induces an inclusion:

$$\mathrm{Hom}_{\mathsf{F}}\left(\mathsf{E}(a) \otimes S^2, \mathsf{E}(a+2)\right) \to \mathrm{Hom}_{\mathsf{F}}\left(\mathsf{E}(a) \otimes \Lambda^1 \otimes \Lambda^1, \mathsf{E}(a+2)\right).$$

By duality, $\mathrm{Hom}_{\mathcal{F}}(E(a) \otimes \Lambda^1 \otimes \Lambda^1, E(a+2)) \cong \mathrm{Hom}_{\mathcal{F}}(DE(a+2), DE(a) \otimes (\Lambda^1)^{\otimes 2})$. The inclusion $(\Lambda^1)^{\otimes 2} \hookrightarrow \bar{I}^{\otimes 2}$ and an adjunction map argument show that the second Hom-space is of dimension at most one. The product map $I \otimes I \to I$ induces a non-trivial map $E(a) \otimes S^2 \to E(a+2)$, which is therefore the unique such non-trivial map. The restriction of this map $E(a) \otimes \Lambda^1 \to E(a+2)$ is non-trivial; hence the exact sequence of Hom-sets gives the required result.

The inductive step ($a > 1$, $b > 2$) is treated exactly as in the preceding lemma, using the functor Δ. ∎

Proof of Proposition 3.1. For the first statement, suppose that $F \subset \bar{I} \otimes \Lambda^n$ is a sub-functor such that F has only a finite number of composition factors which are not exterior powers. It follows from the fact that the functors are analytic that there is an integer t such that the image $\hat{F}$ of the composite $F \hookrightarrow \bar{I} \otimes \Lambda^n \to (\bar{I}/p_t\bar{I}) \otimes \Lambda^n$ has composition factors entirely consisting of exterior powers.

Consider such a sub-functor $\hat{F} \subset (\bar{I}/p_t\bar{I}) \otimes \Lambda^n$. The exponential property for exterior powers (in its dual formulation) shows that the only exterior power in the socle of $(\bar{I}/p_t\bar{I}) \otimes \Lambda^n$ is Λ^{n+t+1}. Hence, by Lemma 4.2, if $\hat{F}$ is not isomorphic to Λ^{n+t+1}, it contains a sub-object $E(n+t+1)$ or $DE(n+t)$. However, $\mathrm{Hom}_{\mathcal{F}}(DE(n+t), (\bar{I}/p_t\bar{I}) \otimes \Lambda^n) \cong \mathrm{Hom}_{\mathcal{F}}(DE(n+t), \Lambda^{t+1} \otimes \Lambda^n)$, since Λ^{n+t} is not in the socle of $(\bar{I}/p_{t+1}\bar{I}) \otimes \Lambda^n$, so $\mathrm{Hom}_{\mathcal{F}}(DE(n+t), (\bar{I}/p_t\bar{I}) \otimes \Lambda^n) = 0$, again by Lemma 4.2. A similar argument shows that $\mathrm{Hom}_{\mathcal{F}}(E(n+t+1), (\bar{I}/p_t\bar{I}) \otimes \Lambda^n) = 0$, this time using Lemma 4.3 and Lemma 4.2. This proves that $\hat{F}$ is isomorphic to Λ^{n+t+1}; in particular, this implies that F is finite, as required.

For the second statement, if G is a quotient of $\bar{I} \otimes \Lambda^n$, then G is not finite, by Lemma 2.1. If G had only finitely many non-exterior powers as composition factors, then there is a quotient G' which is non-finite and consists entirely of exterior powers (again, since the functors involved are analytic).

Using arguments similar to the above, Lemma 4.2 shows that there must exist a sub-quotient of this map which is of the form $E(m) \otimes \Lambda^n \twoheadrightarrow E(m+n)$. This is not possible, by Lemma 4.4. ∎

APPENDIX: EXACTNESS OF THE COMPLEX $(\bar{I} \otimes \Lambda^n, \bar{\phi}_n)$

There is a natural embedding $\Lambda^{n+1} \hookrightarrow \Lambda^1 \otimes \Lambda^n \hookrightarrow \bar{I} \otimes \Lambda^n$, given by the coproduct map. The composite $\Lambda^{n+1} \hookrightarrow \bar{I} \otimes \Lambda^n \xrightarrow{\bar{\phi}_n} \bar{I} \otimes \Lambda^{n-1}$ is trivial. This gives rise to a short exact sequence of complexes:

$$\begin{array}{ccccccccc}
\to & \Lambda^{n+1} & \xrightarrow{0} & \Lambda^n & \to & \cdots & \to & \Lambda^1 & \to 0 \\
 & \downarrow & & \downarrow & & & & \downarrow & \\
\to & \bar{I} \otimes \Lambda^n & \xrightarrow{\bar{\phi}_n} & \bar{I} \otimes \Lambda^{n-1} & \to & \cdots & \to & \bar{I} & \to 0 \\
 & \downarrow & & \downarrow & & & & \downarrow & \\
\to & (\bar{I} \otimes \Lambda^n)/\Lambda^{n+1} & \to & (\bar{I} \otimes \Lambda^{n-1})/\Lambda^n & \to & \cdots & \to & \bar{I}/\Lambda^1 & \to 0
\end{array}$$

Denote the bottom chain complex by $C_n = (\bar{I} \otimes \Lambda^n)/\Lambda^{n+1}$, for $n \geq 0$; then there is a filtration of the chain complex given by $f_{k+n}(C_n) = (\bar{P}_k \otimes \Lambda^n)/\Lambda^{n+1}$. This allows the calculation of the homology of C_* via a spectral sequence, which collapses at the E^1 term. Namely, for $k \geq 2$, the kth graded complex associated to this filtration is precisely the *Koszul complex* (truncated on the left):

$$0 \to (\Lambda^1 \otimes \Lambda^{k-1})/\Lambda^k \to \cdots \to \Lambda^{k-1} \otimes \Lambda^1 \to \Lambda^k \to 0,$$

which is exact when k is odd and has homology Λ^n appearing in $\Lambda^n \otimes \Lambda^n$ when $k = 2n$. This proves that:

$$H_n(C_*) \cong \begin{cases} \Lambda^{n-1}, & \text{for } n \geq 2 \\ 0, & n = 1. \end{cases}$$

The calculation of the homology of the top row is not difficult (!!), and to prove that the middle row has trivial homology it suffices to show that the boundary map in the associated homology long exact sequence is non-trivial. Intuitively, this is equivalent to showing that the factor of Λ^n appearing in $\Lambda^n \otimes \Lambda^n$ is sent to the factor of Λ^n in the socle of $\bar{I} \otimes \Lambda^{n-1}$.

This may be done explicitly; it suffices to show that there is an element in $(\bar{I} \otimes \Lambda^n)(\mathbb{F}^n)$ which is sent to a generator of $\Lambda^n(\mathbb{F}^n)$ in $\mathrm{soc}(\bar{I} \otimes \Lambda^{n-1})$. Let $x_1, \dots, x_n$ be a basis of $\mathbb{F}^n$ and consider the element Φ of $(\bar{I} \otimes \Lambda^n)(\mathbb{F}^n)$

$$\left\{ \sum_{i=1}^{n} x_i + \sum_{i:[2]\hookrightarrow[n]} x_{i(1)}x_{i(2)} + \cdots + \sum_{i:[j]\hookrightarrow[n]} x_{i(1)} \cdots x_{i(j)} + \cdots + x_1 x_2 \cdots x_n \right\}$$
$$\otimes (x_1 \wedge \cdots \wedge x_n),$$

where $[j]$ denotes the set $\{1, \dots, j\}$ and the product on the left-hand side of the tensor product is given by the product in $\bar{I}$.

The reader may find it amusing to verify that $\bar{\phi}_n(\Phi) = \sum_{i=1}^n x_i \otimes (x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n)$, where the $\hat{}$ indicates the omission of a term (recall that $y^2 = y$ in $\bar{I}$). This proves the required result.

Remark A.1. The author has chosen to sketch the above proof, since he feels it gives an insight as to how the non-exactitude of the Koszul complexes is ''repaired'' when one studies the complex $\bar{I} \otimes \Lambda^n$.

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REFERENCES

1. D. Carlisle and N. J. Kuhn, Subalgebras of the Steenrod algebra and the action of matrices on truncated polynomial algebras, *J. Algebra* **121** (1989), 370–387.
2. V. Franjou, Extensions entre puissances extérieures et entre puissances symétriques, *J. Algebra* **179** (1996), 501–522.
3. V. Franjou, J. Lannes, and L. Schwartz, Autour de la cohomologie de MacLane des corps finis, *Invent. Math.* **115** (1994), 513–538.
4. H.-W. Henn, J. Lannes, and L. Schwartz, The categories of unstable modules and unstable algebras over the Steenrod algebra modulo nilpotent objects, *Amer. J. Math.* **115** (1993), 1053–1106.
5. N. J. Kuhn, Generic representations of the finite general linear groups and the Steenrod algebra, I, *Amer. J. Math.* **116** (1993), 327–360.
6. N. J. Kuhn, Generic representations of the finite general linear groups and the Steenrod algebra, II, *K-Theory* **8** (1994), 395–426.
7. N. J. Kuhn, Generic representations of the finite general linear groups and the Steenrod algebra, III, *K-Theory* **9** (1995), 273–303.
8. L. Piriou, Thèse de doctorat, Université de Paris VII, 1995.
9. G. M. L. Powell, The Artinian conjecture for $I \otimes I$, *J. Pure Appl. Algebra*, in press. [With an Appendix by L. Schwartz]
10. G. M. L. Powell, Polynomial filtrations and Lannes' T-functor, preprint, 1996.
11. G. M. L. Powell, The structure of indecomposable injectives in generic representation theory, preprint, 1996.
12. L. Schwartz, ''Unstable Modules over the Steenrod Algebra and Sullivan's Fixed Point Set Conjecture,'' Chicago Lecture Notes in Mathematics, Univ. of Chicago Press, Chicago/London, 1994.

[1] The earlier version appeared as Prépublication Mathematiques 96-02 de l'Université de Paris-Nord.